

Stability of Equilibria in Modified-Gradient Systems

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Abstract

Motivated by questions in biology, we investigate the stability of equilibria of the dynamical system $\mathbf{x}' = P(t)\nabla f(x)$ which arise as critical points of f , under the assumption that $P(t)$ is positive semi-definite. It is shown that the condition $\int_0^\infty \lambda_1(P(t)) dt = \infty$, where $\lambda_1(P(t))$ is the smallest eigenvalue of $P(t)$, plays a key role in guaranteeing uniform asymptotic stability and in providing information on the basin of attraction of those equilibria.

Keywords: dynamical systems, modified gradient system, equilibria, asymptotic stability, basin of attraction

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1. Introduction

The evolution of continuous phenotypes, for example height, by means of natural selection is a central theme in evolutionary biology. The breeder's equation ($R = h^2s$) was first introduced by Lush in 1937 [1] to predict the change in phenotype (R) with respect to the heritability (h^2) and strength of natural selection (s). In a seminal series of papers, the breeder's equation was updated to the so-called multivariate breeder's equation by Lande [2, 3] and Lande and Arnold [4]. The multivariate breeder's equation is often presented in varying forms such as $\Delta\bar{z}(t) = h^2\sigma^2\partial\ln(\bar{W})/\partial\bar{z}(t)$ [2], $\Delta\bar{z} = \mathbf{G}\nabla\ln(\bar{\mathbf{W}})$ [3], $\Delta\bar{z} = \mathbf{G}\mathbf{P}^{-1}\mathbf{s}$ [4], and $\Delta\bar{z} = \mathbf{G}\beta$ [5], as well as continuous-time counterparts (i.e., $d\bar{z}/dt$); all of these forms reduce to the concept that the change in mean phenotype (\bar{z}) over time is given by the product of a genetic variance-covariance matrix (\mathbf{G}) and the gradient of the logarithm of the average fitness function ($\bar{W}(\bar{z})$). As of December 2015, Web of Science indicates that the papers by Lande [2, 3] and Lande and Arnold [4] have garnered at least 791, 1 442, and 2 852 citations, respectively, which gives some idea of the impact these works have had on evolutionary biology and related fields.

One of the critical assumptions in much of this research is that the so-called \mathbf{G} -matrix is constant. A Web of Science search indicates at least 175 papers on the constancy and form of the \mathbf{G} -matrix with 66 of those published since January 2010 (a broader search on “genetic constraints” reveals many more relevant publications). The principal concern is that the \mathbf{G} -matrix limits how traits evolve and approach their evolutionary optima [3, 6, 7]. For example, Dickerson [8] studied a special case of equal genetic variances which produces a \mathbf{G} -matrix with a zero eigenvalue, thus preventing evolution along some trajectories. Furthermore, Pease and Bull [9] examined “ill-conditioned” \mathbf{G} -matrices where the ratio of the largest to the smallest eigenvalue is large and concluded that the speed of evolution toward an optimum is greatly reduced. Other work has suggested that the number of dimensions in the system affects stability [10]. However, formal criteria for when and how an evolutionary system will converge upon an equilibrium are lacking. While most research considers \mathbf{G} to be constant, it is widely recognized that \mathbf{G} itself is expected to evolve over time [4, 6, 11, 12].

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Considering \mathbf{G} to be time-varying further muddies the waters of whether such systems approach and are stable at existing equilibria and lacks formal mathematical treatment.

Similarly, considerable interest has been paid to rugged fitness landscapes where the average fitness function has multiple peaks (optima) [13–16]. Exploration of fitness landscapes, in other words movement between different optima, is a key part of Wright’s shifting balance theory [17]. Despite interest in which evolutionary optimum the population mean phenotype will evolve toward, conventional wisdom that the nearest optimum is favored or numerical methods are relied upon. In fact, some research has shown that the nearest optimum is not always the one favored by evolution [13]. As in the case of stability analyses, no rigorous analysis of if and when a particular optimum will be evolved toward has been performed.

The contributions of this paper are threefold. First, we rigorously analyze the modified gradient system commonly used to model the evolution of continuous traits for the existence and stability of equilibria. Our research shows that biologists can simply search for the isolated local maxima of a fitness function; these points are guaranteed to be at least uniformly stable. Second, in cases where the smallest eigenvalue of the \mathbf{G} -matrix, $\lambda_1(P(t))$ in our notation, meets the condition $\int_0^\infty \lambda_1(P(t))dt = \infty$, then the equilibrium is guaranteed to be uniformly asymptotically stable. Finally, an understanding of the inverse image under the fitness function f of intervals of the form $(c, f(\bar{x}))$ gives information on the basin of attraction of an equilibrium at \bar{x} . Taken together, these contributions greatly enhance our ability to analyze and understand multivariate phenotypic evolution.

2. Existence and Stability of Equilibria

Let $x = (x_1, x_2, \dots, x_n)$ denote a point in \mathbb{R}^n , and let $\mathbf{x} = [x_1 \ x_2 \ \cdots \ x_n]^T$ be the corresponding $n \times 1$ vector equivalent. We use the Euclidean norm as a measure of distance and we let $B_\delta(\bar{x})$ denote the open ball of radius δ centered at \bar{x} . The object is to determine the stability of equilibrium solutions of the n -dimensional modified-gradient system

$$\mathbf{x}' = P(t)\nabla f(x). \quad (\text{E})$$

Note that the continuous-time multivariate breeder’s equation is of this form with $P(t)$ being the time-dependent \mathbf{G} -matrix, and f being $\ln \bar{W}$. We assume throughout that the following hypothesis holds:

H_0 : D is a domain in \mathbb{R}^n , f is a real-valued C_1 (i.e., continuous with continuous partials) function defined on D , t is nonnegative, the gradient of f denoted by ∇f has components which are C_1 on D , and $P(t)$ is an $n \times n$ matrix-valued function with C_1 -entries that is defined and positive semi-definite for $t \geq 0$.

H_0 guarantees that, for any $t_0 \geq 0$ and any x_0 in D , there is a unique solution of (E) satisfying the initial condition $x(t_0) = x_0$. The assumption that $P(t)$ is positive semi-definite is consistent with biological applications because the \mathbf{G} -matrix is a variance-covariance matrix, and variance-covariance matrices are always symmetric, positive semi-definite matrices.

If f has an isolated maximum value at a point $x = \bar{x}$ of D , then we know from calculus that $\nabla f(\bar{x}) = \mathbf{0}$ so $\mathbf{x} = \bar{\mathbf{x}}$ is an equilibrium (i.e., constant in time) solution of (E). We investigate the stability of such equilibria. Although a translation always allows one to assume the equilibrium point is at $\mathbf{x} = \mathbf{0}$, we will continue to assume, because of our interest in evolutionary applications, that $\mathbf{x} = \bar{\mathbf{x}}$ is the equilibrium solution.

We let $\lambda_1(P(t))$ denote the smallest eigenvalue of $P(t)$ and introduce the eigenvalue condition

$$\int_0^\infty \lambda_1(P(t))dt = \infty. \quad (\text{EC})$$

This condition will play an important role in what follows.

We follow the definitions of uniform stability and uniform asymptotic stability as given by Hartman [18]. In contrast to some definitions, this definition of uniform asymptotic stability gives uniformity in the choice of starting time t_0 and does not involve the rate at which solutions tend to the equilibrium solution. Consider the following additional hypotheses:

- H_1 : f has an isolated local maximum value at the point $\bar{x} \in D$;
- H_2 : \bar{x} is an isolated critical point of f ; and

H_3 : eigenvalue condition (EC) holds.

Our stability results are contained in the following theorem.

Theorem 1 (Stability and Asymptotic Stability).

- (i) If H_0 and H_1 hold, then $x = \bar{x}$ is a uniformly stable equilibrium solution of (E).
- (ii) If H_0 , H_1 , H_2 and H_3 all hold, then $x = \bar{x}$ is a uniformly asymptotically stable equilibrium solution of (E).

Proof. Suppose H_0 and H_1 hold. Let $M = f(\bar{x})$ and define the function V on D by $V(x) = M - f(x)$. For a solution $x(t)$ of (E), let V_x be the function defined by $V_x(t) = V(x(t))$ for t in the interval of existence of $x(t)$. The so-called trajectory derivative is then given by

$$\begin{aligned} V'_x(t) &= \frac{\partial V}{\partial x_1}(x(t))x'_1(t) + \frac{\partial V}{\partial x_2}(x(t))x'_2(t) + \cdots + \frac{\partial V}{\partial x_n}(x(t))x'_n(t) \\ &= \mathbf{x}'(t) \cdot \nabla V(x(t)) = -(P(t)\nabla f(x(t))) \cdot \nabla f(x(t)). \end{aligned}$$

So, $V(\bar{x}) = 0$ and, in an appropriately chosen neighborhood of \bar{x} , $V(x) > 0$ for $x \neq \bar{x}$ and the trajectory derivatives are nonpositive since $P(t)$ is positive semi-definite. By standard Lyapunov theory (see, for example, Theorem 8.3, p. 40 of Hartman [18]), $x = \bar{x}$ is a uniformly stable equilibrium solution of (E). This proves (i).

Now assume H_0 , H_1 , H_2 and H_3 all hold. Let M and V be defined as in the proof of (i). Suppose $\varepsilon > 0$ is given. Since \bar{x} is an interior point of D and since H_1 and H_2 hold, we can restrict $\varepsilon > 0$ to be so small that $x \in D$, $f(x) < M$ and $\nabla f(x) \neq \mathbf{0}$ for $0 < |x - \bar{x}| \leq \varepsilon$. Since we have uniform stability by Part (i), we find $\delta > 0$ with $\delta < \varepsilon$ such that any solution $x(t)$ satisfying $|x(t_0) - \bar{x}| < \delta$ at some time $t_0 \geq 0$ also satisfies $|x(t) - \bar{x}| < \varepsilon$ for all $t \geq t_0$.

Let $x(t)$ be any solution with $|x(t_0) - \bar{x}| < \delta$ at some time $t_0 \geq 0$. To complete the proof of Part (ii), we need to show that $\lim_{t \rightarrow \infty} x(t) = \bar{x}$. Since we have uniqueness of solutions to initial value problems, we can assume $x(t) \neq \bar{x}$ for $t \geq t_0$. Also, for $t \geq t_0$, $P(t)$ is positive semi-definite and $|x(t) - \bar{x}| < \varepsilon$ so we have $V_x(t) = M - f(x(t)) > 0$ and $V'_x(t) = -(P(t)\nabla f(x(t))) \cdot \nabla f(x(t)) \leq 0$. Therefore, $c = \lim_{t \rightarrow \infty} V_x(t)$ exists with $c \geq 0$. We prove that $c = 0$. Suppose not, then $c > 0$. Since $V(\bar{x}) = 0$, we use the continuity of V to choose δ_1 with $0 < \delta_1 < \delta$ so that $V(x) < c$ for $|x - \bar{x}| < \delta_1$. Because $V_x(t) = V(x(t)) > c$ for all $t \geq t_0$, the trajectory $x(t)$ stays in the region $\{x : \delta_1 \leq |x - \bar{x}| \leq \varepsilon\}$ for all $t \geq t_0$. On this compact region, the continuous function $\nabla f(x) \cdot \nabla f(x)$ is positive and hence assumes a positive minimum value m_1 at some point in the set. Hence, for $t \geq t_0$, we get that

$$V'_x(t) \leq -\lambda_1(P(t))\nabla f(x(t)) \cdot \nabla f(x(t)) \leq -m_1\lambda_1(P(t)).$$

Consequently,

$$V_x(t) = V_x(t_0) + \int_{t_0}^t V'_x(s)ds \leq V_x(t_0) - \int_{t_0}^t m_1\lambda_1(P(s))ds$$

for $t \geq t_0$. But then (EC) implies that $V_x(t) \rightarrow -\infty$ as $t \rightarrow \infty$ contradicting that $V_x(t)$ stays positive. This proves that $\lim_{t \rightarrow \infty} V_x(t) = 0$.

Finally, we prove that $\lim_{t \rightarrow \infty} x(t) = \bar{x}$. Suppose not. Then there exists ε_1 with $0 < \varepsilon_1 < \varepsilon$ and arbitrarily large values of t where $|x(t) - \bar{x}| \geq \varepsilon_1$. For such t , $V_x(t) = V(x(t)) \geq m_2$ where m_2 is defined to be the minimum value of $V(x)$ on the compact set $\{x : \varepsilon_1 \leq |x - \bar{x}| \leq \varepsilon\}$. Because $m_2 > 0$, this contradicts $\lim_{t \rightarrow \infty} V_x(t) = 0$ completing the proof. \square

Example 2.1 Asymptotic Stability requires Eigenvalue Condition. The following example illustrates that an eigenvalue condition, such as we have given in (EC), is necessary in order to obtain asymptotic stability. Let $P(t) = \begin{bmatrix} (t+1)^{-2} & 0 \\ 0 & (t+1)^{-1} \end{bmatrix}$ and $f(x_1, x_2) = 4 - (x_1 - 1)^2 - (x_2 - 1)^2$ and consider the

associated system $\mathbf{x}' = P(t)\nabla f(x)$ for $t \geq 0$. Then $\lambda_1(P(t)) = (t+1)^{-2}$ and $\lambda_2(P(t)) = (t+1)^{-1}$, therefore $\int_0^\infty \lambda_1(P(t))dt < \infty$ while $\int_0^\infty \lambda_2(P(t))dt = \infty$. Thus, Theorem 1.i holds ($x_1(t) \equiv x_2(t) \equiv 1$ is a uniformly stable equilibrium solution) while Theorem 1.ii does not. The closed form solution is given by $x_1(t) = 1 + c_1 \exp(2/(t+1))$, $x_2(t) = 1 + c_2(t+1)^{-2}$ for arbitrary constants c_1 and c_2 . We see that $\lim_{t \rightarrow \infty} x_1(t) = 1 + c_1 \neq 1$; hence the equilibrium solution ($x_1(t) \equiv 1$) is not asymptotically stable.

More specifically, if f has an isolated local maximum at \bar{x} but \bar{x} is not an isolated critical point, then we can only conclude stability, not asymptotic stability.

Example 2.2 Asymptotic Stability requires Isolated Critical Point. Here we produce an example which shows that the hypothesis H_2 is essential to the conclusion that $\mathbf{x} = \bar{\mathbf{x}}$ is asymptotically stable when $P(t)$ satisfies (EC) as in Theorem 1.ii. Even for gradient systems, the necessity of adding the assumption that the point where the isolated local extremum occurs is also an isolated critical point has been missed by some authors (e.g., Part 3 of the theorem on page 205 of Hirsch et al. [19]).

We create a radially symmetric function $f(r, \theta)$ using polar coordinates that is continuously differentiable on the unit circle $r \leq 1$, that has an absolute maximum value at the origin, that decreases as r increases, and is such that there is a sequence of concentric circles $r = r_i$ with r_i decreasing to zero as $i \rightarrow \infty$ and with each $r = r_i$ consisting entirely of critical points of f . We again let $P(t)$ be the 2×2 identity matrix, thus satisfying (EC). The system $\mathbf{x}' = P(t)\nabla f$ will then have the properties we seek, namely, we no longer have isolated critical points of f .

We first define sequences x_n and z_n by $x_n = -2^{-n}$ and $z_n = (1 - 4^{-n})/3$ for $n = 0, 1, 2, \dots$. Let I_n be the interval $[x_n, x_{n+1}]$. The union of the intervals I_n is then the interval $[-1, 0)$. We define a function $p(x)$ on the interval $[-1, 0)$ which restricted to the interval I_n is a cubic polynomial $p_n(x)$. Furthermore, we require each $p_n(x)$ to satisfy

$$p_n(x_n) = z_n, \quad p'_n(x_n) = 0, \quad p_n(x_{n+1}) = z_{n+1}, \quad p'_n(x_{n+1}) = 0. \quad (1)$$

Letting $p_n(x) = \alpha(x - x_n)^3 + \beta(x - x_n)^2 + \gamma(x - x_n) + \delta$ and using the requirements in (1), we find after some algebra and calculus that $\alpha = -2^{n+2}$, $\beta = 3$, $\gamma = 0$ and $\delta = (1 - 4^{-n})/3$. We then find, again using calculus, that $p'_n(x) > 0$ for x in the open interval (x_n, x_{n+1}) and the maximum value of p'_n on the interval I_n is $3/2^{n+2}$. We then extend $p(x)$ to the closed interval $[-1, 0]$ by defining $p(0) = 1/3$. This makes p continuous on $[-1, 0]$. Considering difference quotients, it is easy to see that the left-hand derivative of p at $x = 0$ exists and has value zero. We symmetrically extend the definition of p to the interval $[-1, 1]$ by letting $p(x) = p(-x)$ for $0 < x \leq 1$. Taking into account the way the cubic polynomials were pieced together at the endpoints and the fact that the maximum value of $p'(x)$ on the interval I_n approaches zero as $n \rightarrow \infty$, we see that p has a continuous derivative on the interval $[-1, 1]$.

Finally, we define the radially symmetric $f(r, \theta) = f(r)$ by taking $f(r) = p(r)$ for $0 \leq r \leq 1$, $0 \leq \theta \leq 2\pi$. Clearly, at any point on a circle $r = |x_n|$, we have $f_r = f_\theta = 0$ since $p'(x_n) = 0$ and f is independent of θ . Hence, all points on $r = |x_n|$ are critical points of f and yield equilibrium solutions of $\mathbf{x}' = \nabla f$. Even though f has an isolated maximum value at the origin, $x = (0, 0)$ is not an asymptotically stable equilibrium solution since solutions starting at $t = 0$ between two concentric circles $r = |x_n|$ and $r = |x_{n+1}|$ are trapped in that region and cannot approach the origin as $t \rightarrow \infty$. Of course, Theorem 1.i still applies to give that $x = (0, 0)$ is a stable equilibrium.

3. Basin of Attraction

Given a uniformly asymptotically stable equilibrium \bar{x} of (E), it is of interest to know the set of points x_0 such that the trajectory starting at point x_0 at some time t_0 exists for all $t \geq t_0$ and approaches \bar{x} as t tends to infinity; that is, the so-called *basin of attraction* of \bar{x} . The following theorem provides information on the basin of attraction in the setting of Theorem 1.ii.

Theorem 2 (Basin of Attraction). *Suppose H_0 , H_1 , H_2 and H_3 all hold. Let $M = f(\bar{x})$, let c be a real number less than M and let $O_{c, \bar{x}}$ be the set defined by $O_{c, \bar{x}} = \{\bar{x}\} \cup \{x : c < f(x) < M\}$. Then $O_{c, \bar{x}}$ is open and has a unique component $E_{c, \bar{x}}$ that contains \bar{x} . Let $\partial E_{c, \bar{x}}$ denote the boundary of $E_{c, \bar{x}}$, and let*

$\overline{E_{c,\bar{x}}} = E_{c,\bar{x}} \cup \partial E_{c,\bar{x}}$ denote the closure of $E_{c,\bar{x}}$. Consider additional hypotheses:

H_4 : $E_{c,\bar{x}}$ is bounded and $\overline{E_{c,\bar{x}}}$ is contained in D ;

H_5 : $f(x) = c$ for all x in $\partial E_{c,\bar{x}}$; and

H_6 : f has no critical points other than \bar{x} in $\overline{E_{c,\bar{x}}}$.

If H_4 , H_5 , and H_6 also hold, then $E_{c,\bar{x}}$ is contained in the basin of attraction of \bar{x} .

Proof. Suppose H_0 through H_6 all hold. Let M , c and $O_{c,\bar{x}}$ be as defined above. We first prove $O_{c,\bar{x}}$ is open. By the continuity of f , the set $\{x : c < f(x) < M\}$ is open. Using the continuity of f and the fact that f has an isolated maximum value at \bar{x} , choose $\delta > 0$ such that f is defined on the ball $B_\delta(\bar{x})$, $f(x) < M$ for $0 < |x - \bar{x}| < \delta$ and $|f(x) - M| < M - c$ for $x \in B_\delta(\bar{x})$. Then $c < f(x) < M$ for $0 < |x - \bar{x}| < \delta$ so $B_\delta(\bar{x})$ is open, contains \bar{x} , and is contained in $O_{c,\bar{x}}$. It follows that the set $O_{c,\bar{x}} = B_\delta(\bar{x}) \cup \{x : c < f(x) < M\}$ is open. Hence, there is a unique open component $E_{c,\bar{x}}$ of $O_{c,\bar{x}}$ that contains the point \bar{x} .

Let x_0 be any point of $E_{c,\bar{x}}$ and let $x(t)$ be the solution of (E) satisfying the initial condition $x(t_0) = x_0$ for some $t_0 \geq 0$. We wish to prove that $x(t)$ exists for $t \geq t_0$ and $\lim_{t \rightarrow \infty} x(t) = \bar{x}$. This is clearly true if $x_0 = \bar{x}$ so we assume $x_0 \neq \bar{x}$ and, in light of the uniqueness of solutions to initial value problems, that $x(t) \neq \bar{x}$ for all $t \geq t_0$. As before, we let $V(x) = M - f(x)$ for x in D . While the trajectory $x(t)$ remains in $E_{c,\bar{x}}$, we have by H_6 that the trajectory derivative satisfies $V'_x(t) = -P(t)\nabla f(x(t)) \cdot \nabla f(x(t)) < 0$. Because $V_x(t_0) < M - c$ and $V_x(t)$ decreases as t increases, the trajectory $x(t)$ can never reach $\partial E_{c,\bar{x}}$ where, by H_5 , $V_x(t)$ would equal $M - c$. Hence, $x(t)$ stays in the region $E_{c,\bar{x}}$ and therefore in the set $\overline{E_{c,\bar{x}}}$ so long as the solution $x(t)$ continues to exist. By H_4 , $\overline{E_{c,\bar{x}}}$ is both closed and bounded and therefore compact. Since $x(t)$ stays in a compact subset of D , it follows directly from Theorem 3.1 of Hartman [18], that the right-maximal interval of existence of $x(t)$ as a solution of (E) cannot be of the form $[t_0, \omega)$ with $\omega < \infty$. Thus, the solution $x(t)$ exists for all $t \geq t_0$.

From here on, the proof essentially follows that of Theorem 1.ii, but we repeat some of the details for clarity. First, let $\lim_{t \rightarrow \infty} V_x(t) = \alpha$ and suppose $\alpha > 0$. Then using the continuity of V , find $\delta > 0$ small enough that $0 < V(x) < \alpha$ for $0 < |x - \bar{x}| < \delta$. Now the set $\overline{E_{c,\bar{x}}} \setminus B_\delta(\bar{x})$ is closed and bounded by H_4 , so, by H_6 , the continuous function $\nabla f(x) \cdot \nabla f(x)$ assumes a positive minimum m_1 on that set. Because $x(t)$ never enters the set $B_\delta(\bar{x})$ where we would have $V_x(t) = V(x(t)) < \alpha$, we get that $V'_x(t) \leq -m_1 \lambda_1(P(t))$ for $t \geq t_0$. This leads to $V_x(t) \rightarrow -\infty$ as $t \rightarrow \infty$, a contradiction which shows that $\alpha = 0$.

The next step is to prove that $\lim_{t \rightarrow \infty} x(t) = \bar{x}$. To do this, suppose $\lim_{t \rightarrow \infty} x(t) \neq \bar{x}$. There then exists an $\varepsilon > 0$ such that $|x(t) - \bar{x}| \geq \varepsilon$ for arbitrarily large values of t . The function $V(x)$ is positive and continuous on the compact set $\overline{E_{c,\bar{x}}} \setminus B_\varepsilon(\bar{x})$, hence, $V(x)$ has a positive minimum, call it m_2 , on the set $\overline{E_{c,\bar{x}}} \setminus B_\varepsilon(\bar{x})$. However, there are arbitrarily large values of t where $x(t) \in \overline{E_{c,\bar{x}}} \setminus B_\varepsilon(\bar{x})$ for which $V_x(t) = V(x(t)) \geq m_2$. This contradicts $\lim_{t \rightarrow \infty} V_x(t) = 0$ and completes the proof. \square

We note that LaSalle's Theorem can be used to obtain information on the basin of attraction of an equilibrium solution—for examples see Theorem 6.1 in Leighton [20], Theorem 11.11 in Miller and Michel [21], or the theorem on p. 200 of Hirsch et al. [19]. However, those results deal with autonomous systems and do not apply to (E).

Example 3.1 Basins of Attraction. We conclude by giving an example illustrating both the use of Theorem 2 and the role played by hypotheses H_5 and H_6 of that theorem. Let $f(x_1, x_2) = 96x_2 - 84x_2^2 + 28x_2^3 - 3x_2^4 - 10(x_1 - 2)^2$. Let $P(t)$ be any 2×2 matrix-valued function defined and continuous for $t \geq 0$ and such that the eigenvalue condition (EC) holds. Equation (E) becomes

$$\begin{bmatrix} x'_1 \\ x'_2 \end{bmatrix} = P(t) \begin{bmatrix} -20(x_1 - 2) \\ -12(x_2 - 1)(x_2 - 2)(x_2 - 4) \end{bmatrix}.$$

Then f has local maximum values at the points $p_1 = (2, 1)$ and $p_2 = (2, 4)$ and a saddle at $p_3 = (2, 2)$ with $f(2, 1) = 37$, $f(2, 4) = 64$ and $f(2, 2) = 32$.

For a real number c , let L_c denote the level set defined by $L_c = \{(x_1, x_2) : f(x_1, x_2) = c\}$. L_{33} consists of two simple closed curves; we let C_{p_1} and C_{p_2} denote the curve having the point p_1 and p_2 (respectively) as an interior point. Then the set E_{33, p_1} consists of all points interior to C_{p_1} while E_{33, p_2} consists of all points interior to C_{p_2} . Theorem 2 applies and shows that all trajectories $x(t)$ having $x(t_0)$ in

E_{33,p_1} tend to p_1 as $t \rightarrow \infty$, with a similar conclusion for trajectories in E_{33,p_2} . It is interesting to consider $E_{20,p_1} = \{p_1\} \cup \{x : 20 < f(x) < 37\}$ and $E_{20,p_2} = \{p_2\} \cup \{x : 20 < f(x) < 64\}$. First, E_{20,p_2} contains all points interior to a simple closed curve containing both p_1 and p_2 in its interior; hence, Theorem 2 does not apply to E_{20,p_2} because H_6 is violated. On the other hand, E_{20,p_1} consists of $E_{20,p_2} \setminus \overline{E_{37,p_2}}$. Now, Theorem 2 does not apply to E_{20,p_1} because the boundary of E_{20,p_1} contains points of the level set L_{37} at which f takes on the value 37 thus violating H_5 ; clearly some trajectories starting in E_{20,p_1} will tend toward the boundary points in L_{37} while others will tend toward p_1 .

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